Mirror Symmetry of Calabi-Yau Manifolds and Flat Coordinates

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Abstract

We study mirror symmetry of Calabi-Yau manifolds within the framework of the Gauss-Manin system. Applying the flat coordinates to the Gauss-Manin system for the periods, we derive differential equations for the mirror map in addition to the ordinary Picard-Fuchs equations for the periods. These equations are obtained for a class of one-parameter models and a two-parameter model of Fermat type Calabi-Yau manifolds.

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1 Introduction

In string theory, one of the most important problems is to derive a realistic low-energy theory. To discuss low-energy theories, one has to determine masses, gauge coupling constants, etc. These various couplings depend on the moduli of the string model. In the context of conformal field theory, the moduli are described by the marginal operators. When we consider the Calabi-Yau compactifications down to four dimensions, these are identified as the harmonic $(1,1)$ and $(2,1)$ forms which describe the Kähler and complex structure deformations, respectively. The number of the deformation parameters is equal to the Hodge numbers $h_{1,1}$ and $h_{2,1}$. Since there are no couplings between the two different types of deformations, the Yukawa couplings are split into two sectors. It is well-known that the Yukawa couplings corresponding to complex structure deformations receive no loop collections. On the other hand, the Yukawa couplings corresponding to Kähler deformations are renormalized by instantons [1] [2] [3]. Therefore at first glance one can analyze half of the moduli dependence in the Calabi-Yau compactifications exactly.

It was, however, discovered by Candelas et al. [4] that a mirror symmetry of the Calabi-Yau manifolds enables us to determine the full dependence of Yukawa couplings on the moduli. A mirror symmetry relates two topologically distinct Calabi-Yau manifolds M and $\bar{M}$ which have the relations $h_{1,1} = \bar{h}_{2,1}$ and $h_{2,1} = \bar{h}_{1,1}$ with h and $\bar{h}$ being the Hodge numbers of M and $\bar{M}$, respectively. A more important property of this symmetry is to give isomorphism between the quantum cohomology ring of M and $\bar{M}$ [5]. So, by virtue of this symmetry, one can derive the exact dependence of Yukawa couplings on Kähler deformations from tree level informations of mirror manifold [4] [6] [7] [8] [9] [10] [11]. In these calculations, the crucial point is to construct the mirror map explicitly which relates a mirror pair of Calabi-Yau manifolds. Usually this is done by making a monodromy analysis of periods in the large radius limit.

Recently, Hosono and Lian have clarified that the mirror map is intimately related to

the notion of flat coordinates of the period integrals [12]. The idea of the flat coordinates [13] has found extensive applications, for example, off-critical deformations of $N = 2$ superconformal theories in $2D$ [14], $2D$ quantum gravity and topological field theories [15] [16] [17] [18] [19] [20] [21].

In the context of mirror symmetry of Calabi-Yau manifolds, however, it was tantalizing to see how the flat coordinates play a role when the mirror map is considered. In [12], taking the quintic Calabi-Yau manifold as an example, the flat coordinates are introduced to define a natural basis of the quantum cohomology ring. It is then observed that the flat coordinates are related to the solution of the Picard-Fuchs equations at the point of maximally unipotent monodromy. All this is done in the framework of the Gauss-Manin system which is a set of the first order differential equations for the period integrals.

In this article, we extend the analysis of [12] by taking more examples of Calabi-Yau manifolds with one Kähler modulus and two Kähler moduli. It will be shown that differential equations which govern the periods and the mirror map are naturally obtained by applying the flat coordinates to the Gauss-Manin system.

In section 2, employing the Gauss-Manin differential equations in the flat coordinate system, we analyze a class of one-parameter models which have been studied in [8]. In section 3, the flat coordinate method is applied to a two-parameter model. We study the model defined on the weighted projective space $\mathbf{P}(1, 1, 2, 2, 2)$ whose mirror map has been investigated in [9] [10]. Finally, section 4 is devoted to our conclusions.

2 Flat coordinates for one-parameter models

We first consider the simplest Calabi-Yau manifolds which have the Fermat type defining relation such that $W = \sum_{i=0}^{4} x_i^{n_i}$ with a one-parameter deformation of Kähler class, i.e. $h_{1,1} = 1$. As is listed in [8], these models are characterized by the integer k with $k = 5, 6, 8$

and 10. The defining relations are given by

$$
\begin{aligned}
&k = 5: \\
&M = \{x_i \in \mathbf{P}(1,1,1,1,1) | W_5 = \tfrac{1}{5}(x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5) = 0\}, \\
&k = 6: \\
&M = \{x_i \in \mathbf{P}(2,1,1,1,1) | W_6 = \tfrac{1}{3}x_0^3 + \tfrac{1}{6}(x_1^6 + x_2^6 + x_3^6 + x_4^6) = 0\}, \\
&k = 8: \\
&M = \{x_i \in \mathbf{P}(4,1,1,1,1) | W_8 = \tfrac{1}{2}x_0^2 + \tfrac{1}{8}(x_1^8 + x_2^8 + x_3^8 + x_4^8) = 0\}, \\
&k = 10: \\
&M = \{x_i \in \mathbf{P}(5,2,1,1,1) | W_{10} = \tfrac{1}{2}x_0^2 + \tfrac{1}{5}x_1^5 + \tfrac{1}{10}(x_2^{10} + x_3^{10} + x_4^{10}) = 0\}.
\end{aligned}
\tag{2.1}
$$

These models have first appeared in [22] and possess the Hodge numbers $h_{1,1} = 1$ and

$h_{2,1} = 101, 103, 149$ and 145 for $k = 5, 6, 8$ and 10, respectively.

To get the mirror manifolds, one must consider the orbifoldization of these models by

dividing out a certain discrete symmetry G of the defining relation. There is a deformation

of the defining relations (2.1) that preserves invariance under this discrete symmetry. We

have the deformed polynomials

$$
W_k(\alpha) = W_k - \alpha x_0 x_1 x_2 x_3 x_4,
\tag{2.2}
$$

where α is a deformation parameter. Let M_α be the manifold defined by $W_k(\alpha) = 0$, then

its mirror manifold $\bar{M}_\alpha$ is obtained as

$$
\bar{M}_\alpha = M_\alpha / G,
\tag{2.3}
$$

where G is $\mathbf{Z}_5^3$, $\mathbf{Z}_3 \times \mathbf{Z}_6^2$, $\mathbf{Z}_8^2 \times \mathbf{Z}_2$ and $\mathbf{Z}_{10}^2$ for $k = 5, 6, 8$ and 10, respectively. The

deformation by α can now be regarded as the complex structure deformation of the

orbifold (mirror manifold). Actually, by direct calculations on the manifolds [8], one

can see that these manifolds $\bar{M}_\alpha$ have the Hodge numbers $h_{1,1} = 101, 103, 149, 145$ and

$h_{2,1} = 1$, and hence these are mirrors of M's.

In order to discuss the quantum cohomology ring of $\bar{M}_\alpha$ using the Gauss-Manin system,

we introduce the deformed Jacobian ring J_α by

$$
J_\alpha \equiv \mathbf{C}[x_0, \ldots, x_4]^G / (\partial W_k(\alpha)/\partial x_i) \cap \mathbf{C}[x_0, \ldots, x_4]^G.
\tag{2.4}
$$

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We fix a basis of J_α as $\{1, \phi, \phi^2, \phi^3\}$ where $\phi \equiv x_0 x_1 x_2 x_3 x_4$. Let us define the period integrals $w_{\gamma_j}^{(i)}$ as

$$w_{\gamma_j}^{(i)} = i! \int_{\gamma_j} \frac{\phi^i}{W_k^{i+1}(\alpha)} d\mu, \quad i, j = 0, 1, 2, 3, \tag{2.5}$$

where $d\mu = \sum_{l=0}^{4}(-1)^{l+1} x_l dx_0 \wedge \cdots \wedge \hat{dx}_l \wedge \cdots \wedge dx_4$ and γ_j's are the homology cycles in $H_3(\bar{M}_\alpha, \mathbf{Z})$. Using the relations obtained from $\partial W_\alpha / \partial x_i = \ldots$ and integrating by parts, we derive a set of first order differential equations which are satisfied by the periods (2.5)

$$\frac{\partial}{\partial \alpha} w = G_k(\alpha) w, \tag{2.6}$$

where $w = {}^t(w_{\gamma_j}^{(0)}, w_{\gamma_j}^{(1)}, w_{\gamma_j}^{(2)}, w_{\gamma_j}^{(3)})$ and

$$G_5(\alpha) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\alpha}{1-\alpha^5} & \frac{15\alpha^2}{1-\alpha^5} & \frac{25\alpha^3}{1-\alpha^5} & \frac{10\alpha^4}{1-\alpha^5} \end{pmatrix},$$

$$G_6(\alpha) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\alpha^2}{1-\alpha^6} & \frac{15\alpha^3}{1-\alpha^6} & \frac{25\alpha^6-2}{\alpha^2(1-\alpha^6)} & \frac{10\alpha^6+2}{\alpha(1-\alpha^6)} \end{pmatrix},$$

$$G_8(\alpha) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\alpha^4}{1-\alpha^8} & \frac{15(\alpha^8+1)}{\alpha^3(1-\alpha^8)} & \frac{5(5\alpha^8-3)}{\alpha^2(1-\alpha^8)} & \frac{2(5\alpha^8+3)}{\alpha(1-\alpha^8)} \end{pmatrix},$$

$$G_{10}(\alpha) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\alpha^6}{1-\alpha^{10}} & \frac{5(3\alpha^{10}+7)}{\alpha^3(1-\alpha^{10})} & \frac{5(5\alpha^{10}-7)}{\alpha^2(1-\alpha^{10})} & \frac{10(\alpha^{10}+1)}{\alpha(1-\alpha^{10})} \end{pmatrix}. \tag{2.7}$$

These are the Gauss-Manin system for the periods. These first order systems reduce to the ordinary differential equations for $w_{\gamma_j}^{(0)}$ which take the forms

$$k = 5 \; : \; (1-\alpha^5)w^{(0)''''} - 10\alpha^4 w^{(0)'''} - 25\alpha^3 w^{(0)''} - 15\alpha^2 w^{(0)'} - \alpha w^{(0)} = 0,$$

$$k = 6 \; : \; \alpha^2(1-\alpha^6)w^{(0)''''} - 2\alpha(5\alpha^6+1)w^{(0)'''}$$
$$- (25\alpha^6 - 2)w^{(0)''} - 15\alpha^5 w^{(0)'} - \alpha^4 w^{(0)} = 0,$$

$$k = 8 \; : \; \alpha^3(1-\alpha^8)w^{(0)''''} - 2\alpha^2(5\alpha^8+3)w^{(0)'''}$$

$$-5\alpha(5\alpha^8 - 3)w^{(0)''} - 15(\alpha^8 + 1)w^{(0)'} - \alpha^7 w^{(0)} = 0,$$

$$k = 10 \quad : \quad \alpha^3(1 - \alpha^{10})w^{(0)''''} - 10\alpha^2(\alpha^{10} + 1)w^{(0)'''}$$

$$-5\alpha(5\alpha^{10} - 7)w^{(0)''} - 5(3\alpha^{10} + 7)w^{(0)'} - \alpha^9 w^{(0)} = 0 \tag{2.8}$$

where $' = \partial/\partial\alpha$. These are nothing but the Picard-Fuchs equations [8].

Let us now introduce the flat coordinates and rewrite the Gauss-Manin system. For this we first make a degree-preserving change of basis of the Jacobian ring

$$\{1, \phi, \phi^2, \phi^3\} \to \{\mathcal{O}^{(0)}, \mathcal{O}_t^{(1)}, \mathcal{O}^{t(2)}, \mathcal{O}^{(3)}\} \tag{2.9}$$

with $\mathcal{O}^{(0)} = 1$. Here we regard α as a function of the flat coordinate t ($\alpha = \alpha(t)$). For this basis, the period integrals are defined by

$$v_{\gamma_j}^{(i)} = (-1)^i i! \int_{\gamma_j} \frac{\mathcal{O}^{(i)}}{(r_{11} W_k(\alpha))^{i+1}} d\mu, \tag{2.10}$$

where we have changed the normalization of the defining relation by a certain function $r_{11}(\alpha)$. The period integrals (2.10) are related to (2.5) through

$$w(\alpha) = M_k(\alpha)v(t), \tag{2.11}$$

where $w(\alpha) = {}^t(w^{(0)}, w^{(1)}, w^{(2)}, w^{(3)})$, $v(t) = {}^t(v^{(0)}, v^{(1)}, v^{(2)}, v^{(3)})$ and $M_k(\alpha)$ is a lower-triangular matrix

$$M_k(\alpha) = \begin{pmatrix} r_{11} & 0 & 0 & 0 \\ r_{21} & r_{22} & 0 & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ r_{41} & r_{42} & r_{43} & r_{44} \end{pmatrix}. \tag{2.12}$$

Substituting (2.11) into (2.6) and $\alpha \to t = t(\alpha)$, we see that the Gauss-Manin system for v becomes

$$\frac{\partial}{\partial t} v = \left(\frac{\partial\alpha}{\partial t}\right)\left(M_k^{-1} G_k M_k - M_k^{-1} \frac{\partial M_k}{\partial\alpha}\right) v. \tag{2.13}$$

The basis $\{\mathcal{O}^{(0)}, \mathcal{O}_t^{(1)}, \mathcal{O}^{t(2)}, \mathcal{O}^{(3)}\}$ is constrained to satisfy the flat coordinate conditions

$$\text{i)} \quad \mathcal{O}_t^{(1)} = \frac{\partial}{\partial t}(r_{11} W),$$

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$$\text{ii)} \quad \mathcal{O}_t^{(1)}\mathcal{O}_t^{(1)} = K_{ttt}(t)\mathcal{O}^{t(2)},$$

$$\text{iii)} \quad \mathcal{O}_t^{(1)}\mathcal{O}^{t(2)} = \mathcal{O}^{(3)},$$

$$\text{iv)} \quad \mathcal{O}\mathcal{O}^{(3)} = 0, \quad (\mathcal{O} = \mathcal{O}_t^{(1)}, \mathcal{O}^{t(2)}). \tag{2.14}$$

Then we have

$$\frac{\partial}{\partial t}v = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & K_{ttt}(t) & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}v. \tag{2.15}$$

Comparing (2.13) with (2.15) we determine the transformation matrix M_k. After some algebra we obtain

$$
M_5 = \begin{pmatrix}
r_{11} & 0 & 0 & 0 \\
r'_{11} & r_{11}t' & 0 & 0 \\
r''_{11} & r_{11}t'' + 2r'_{11}t' & \frac{1}{r_{11}t'}\frac{C}{1-\alpha^5} & 0 \\
r'''_{11} & r_{11}t''' + 3r'_{11}t'' + 3r''_{11}t' & \frac{C}{r_{11}t'}\{(\frac{1}{1-\alpha^5})' + \frac{r'_{11}}{r_{11}}\frac{1}{1-\alpha^5}\} & \frac{1}{r_{11}}\frac{C}{1-\alpha^5}
\end{pmatrix},
$$

$$
M_6 = \begin{pmatrix}
r_{11} & 0 & 0 & 0 \\
r'_{11} & r_{11}t' & 0 & 0 \\
r''_{11} & r_{11}t'' + 2r'_{11}t' & \frac{1}{r_{11}t'}\frac{C\alpha}{1-\alpha^6} & 0 \\
r'''_{11} & r_{11}t''' + 3t'_{11}t'' + 3r''_{11}t' & \frac{C}{r_{11}t'}\{(\frac{\alpha}{1-\alpha^6})' + \frac{r'_{11}}{r_{11}}\frac{\alpha}{1-\alpha^6}\} & \frac{1}{r_{11}}\frac{C\alpha}{1-\alpha^6}
\end{pmatrix},
$$

$$
M_8 = \begin{pmatrix}
r_{11} & 0 & 0 & 0 \\
r'_{11} & r_{11}t' & 0 & 0 \\
r''_{11} & r_{11}t' + 2r'_{11}t' & \frac{1}{r_{11}t'}\frac{C\alpha^3}{1-\alpha^8} & 0 \\
r'''_{11} & r_{11}t''' + 3r'_{11}t'' + 3r''_{11}t' & \frac{C}{r_{11}t'}\{(\frac{\alpha^3}{1-\alpha^8})' + \frac{r'_{11}}{r_{11}}\frac{\alpha^3}{1-\alpha^8}\} & \frac{1}{r_{11}}\frac{C\alpha^3}{1-\alpha^8}
\end{pmatrix},
$$

$$
M_{10} = \begin{pmatrix}
r_{11} & 0 & 0 & 0 \\
r'_{11} & r_{11}t' & 0 & 0 \\
r''_{11} & r_{11}t'' + 2r'_{11}t' & \frac{1}{r_{11}t'}\frac{C\alpha^5}{1-\alpha^{10}} & 0 \\
r'''_{11} & r_{11}t''' + 3r'_{11}t'' + 3r''_{11}t' & \frac{C}{r_{11}t'}\{(\frac{\alpha^5}{1-\alpha^{10}})' + \frac{r'_{11}}{r_{11}}\frac{\alpha^5}{1-\alpha^{10}}\} & \frac{1}{r_{11}}\frac{C\alpha^5}{1-\alpha^{10}}
\end{pmatrix},
\tag{2.16}
$$

where $r_{11}(\alpha)$ and $t(\alpha)$ satisfy the following differential equations

$$k=5: \quad r''''_{11} - \frac{10\alpha^4}{1-\alpha^5}r'''_{11} - \frac{25\alpha^3}{1-\alpha^5}r''_{11} - \frac{15\alpha^2}{1-\alpha^5}r'_{11} - \frac{\alpha}{1-\alpha^5}r_{11} = 0,$$

$$t''' - \left(\frac{5\alpha^4}{1-\alpha^5} - \frac{2r'_{11}}{r_{11}}\right)t'' - \left(\frac{5\alpha^3}{1-\alpha^5} + \frac{10\alpha^4}{1-\alpha^5}\frac{r'_{11}}{r_{11}} + \frac{2r'^2_{11}}{r^2_{11}} - \frac{4r''_{11}}{r_{11}}\right)t' = 0,$$

$$k=6: \quad r''''_{11} - \frac{2(5\alpha^6+1)}{\alpha(1-\alpha^6)}r'''_{11} - \frac{(25\alpha^6-2)}{\alpha^2(1-\alpha^6)}r''_{11} - \frac{15\alpha^3}{1-\alpha^6}r'_{11} - \frac{\alpha^2}{1-\alpha^6}r_{11} = 0,$$

$$t''' - \left(\frac{5\alpha^6+1}{\alpha(1-\alpha^6)} - \frac{2r'_{11}}{r_{11}}\right)t'' - \left(\frac{5\alpha^4}{1-\alpha^6} + \frac{10\alpha^6+2}{\alpha(1-\alpha^6)}\frac{r'_{11}}{r_{11}} + \frac{2r'^2_{11}}{r^2_{11}} - \frac{4r''_{11}}{r_{11}}\right)t' = 0,$$

$$k=8: \quad r''''_{11} - \frac{2(5\alpha^8+3)}{\alpha(1-\alpha^8)}r'''_{11} - \frac{5(5\alpha^8-3)}{\alpha^2(1-\alpha^8)}r''_{11} - \frac{15(\alpha^8+1)}{\alpha^3(1-\alpha^8)}r'_{11} - \frac{\alpha^4}{1-\alpha^8} = 0,$$

$$t''' - \left(\frac{5\alpha^8+3}{\alpha(1-\alpha^8)} - \frac{2r'_{11}}{r_{11}}\right)t'' - \left(\frac{5\alpha^8-3}{\alpha^2(1-\alpha^8)} + \frac{10\alpha^8+6}{\alpha(1-\alpha^8)}\frac{r'_{11}}{r_{11}} + \frac{2r'^2_{11}}{r^2_{11}} - \frac{4r''_{11}}{r_{11}}\right)t' = 0,$$

$$k = 10: \qquad r''''_{11} - \frac{10(\alpha^{10}+1)}{\alpha(1-\alpha^{10})}r'''_{11} - \frac{5(5\alpha^{10}-7)}{\alpha^2(1-\alpha^{10})}r''_{11} - \frac{5(3\alpha^{10}+7)}{\alpha^3(1-\alpha^{10})}r'_{11} - \frac{\alpha^6}{1-\alpha^{10}}r_{11} = 0,$$

$$t''' - \left(\frac{5\alpha^{10}+5}{\alpha(1-\alpha^{10})} - \frac{2r'_{11}}{r_{11}}\right)t'' - \left(-\frac{5}{\alpha^2} + \frac{10\alpha^{10}+10}{\alpha(1-\alpha^{10})}\frac{r'_{11}}{r_{11}} + \frac{2r'^2_{11}}{r^2_{11}} - \frac{4r''_{11}}{r_{11}}\right)t' = 0. \quad (2.17)$$

The first equation for each k coincides with the Picard-Fuchs equation (2.8), and the second equation determines the flat coordinate t. Moreover, from eqs.(2.13)$\sim$(2.16), the Yukawa couplings are obtained as

$$K_{ttt} = \frac{1}{r_{11}{}^2}\frac{C\alpha^{k-5}}{1-\alpha^k}\left(\frac{\partial\alpha}{\partial t}\right)^3, \quad k = 5,6,8,10, \qquad (2.18)$$

where C is a constant. This is a desired form in agreement with [4] [8].

The power series solution of the Picard-Fuchs equations (2.8) around $\alpha = \infty$ can be found easily. The solution is given by

$$\bar{w}_0^{(0)}(\alpha) = \frac{1}{\alpha}\sum_{m=0}^{\infty}\frac{\Gamma(km+1)}{\prod_{i=0}^{4}\Gamma(\nu_i m+1)}(\gamma\alpha)^{-km}, \quad |\alpha| > 1, \qquad (2.19)$$

where

$$\nu = \begin{cases} (1,1,1,1,1) & : & k = 5 \\ (2,1,1,1,1) & : & k = 6 \\ (4,1,1,1,1) & : & k = 8 \\ (5,2,1,1,1) & : & k = 10 \end{cases} \qquad (2.20)$$

and $\gamma = k\prod_{i=0}^{4}(\nu_i)^{-\nu_i/k}$. Other solutions $\bar{w}_j^{(0)}(\alpha)$ are generated from (2.19) by

$$\bar{w}_j^{(0)}(\alpha) = \bar{w}_0^{(0)}(\delta^j\alpha), \quad \delta = e^{2\pi i/k}. \qquad (2.21)$$

As a set of fundamental solutions Klemm and Theisen have chosen $\bar{w}_j^{(0)}$ with $j = 0,1,2$ and $k-1$ [8].

On the other hand, it is difficult to deal with the second equations obeyed by t. Let us first examine the asymptotic behavior of t. In the limit $\alpha \to \infty$, eqs.(2.17) turn out to be

$$r''''_{11} + \frac{10}{\alpha}r'''_{11} + \frac{25}{\alpha^2}r''_{11} + \frac{15}{\alpha^3}r'_{11} + \frac{1}{\alpha^4}r_{11} = 0,$$

$$t''' + \left(\frac{5}{\alpha} + \frac{2r'_{11}}{r_{11}}\right)t'' + \left(\frac{5}{\alpha^2} + \frac{10}{\alpha}\frac{r'_{11}}{r_{11}} - \frac{2r'^2_{11}}{r^2_{11}} + \frac{4r''_{11}}{r_{11}}\right)t' = 0 \qquad (2.22)$$

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for $k = 5, 6, 8$ and 10. The solutions of the first equation reads

$$r_{11} \sim \frac{1}{\alpha}, \quad \frac{\log\alpha}{\alpha}, \quad \frac{(\log\alpha)^2}{\alpha}, \quad \frac{(\log\alpha)^3}{\alpha}. \tag{2.23}$$

If we choose the solution $r_{11} \sim 1/\alpha$, then the asymptotic solutions of t are

$$t \sim \log\alpha, \quad (\log\alpha)^2 \tag{2.24}$$

with some appropriate integration constants. Notice that the behavior $t \sim c_1 \log(c_2\alpha)$ coincides with the desired behavior of the mirror map at large radius limit. So, instead of solving them directly, we assume that the solution $t = t(\alpha)$ is given by the mirror map whose explicit form is known to be

$$\begin{aligned}
t(\alpha) &= \frac{2(\bar{w}_1^{(0)} - \bar{w}_0^{(0)}) + \bar{w}_2^{(0)} - \bar{w}_4^{(0)}}{5\bar{w}_0^{(0)}}, \\
t(\alpha) &= \frac{\bar{w}_2^{(0)} - \bar{w}_0^{(0)} + \bar{w}_1^{(0)} - \bar{w}_5^{(0)}}{3\bar{w}_0^{(0)}}, \\
t(\alpha) &= \frac{\bar{w}_2^{(0)} - \bar{w}_0^{(0)} + \bar{w}_1^{(0)} - \bar{w}_7^{(0)}}{2\bar{w}_0^{(0)}}, \\
t(\alpha) &= \frac{\bar{w}_2^{(0)} - \bar{w}_9^{(0)}}{\bar{w}_0^{(0)}}
\end{aligned} \tag{2.25}$$

for $k = 5, 6, 8$ and 10, respectively [8]. Substituting these mirror maps and $r_{11} = \bar{w}_0^{(0)}$ into (2.17) we have checked explicitly that the mirror map (2.25) satisfies (2.17). Thus we observe that the differential equations for the periods and the mirror map are obtained from the Gauss-Manin system in the flat coordinates.

The mirror map (2.25) determines the Yukawa coupling (2.18) up to an arbitrary constant C. This constant is fixed by the well-known fact that the Yukawa coupling in the large radius limit is given by the intersection numbers of a basis for $H^2(M_\alpha, \mathbf{Z})$.

Finally, we present the prepotential within this framework. The Gauss-Manin system (2.15) in the flat coordinates is equivalent to

$$\frac{\partial^2}{\partial t^2} \left(\frac{1}{K_{ttt}(t)} \frac{\partial^2}{\partial t^2} v^{(0)} \right) = 0. \tag{2.26}$$

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For $v^{(0)} = {}^t(v_0^{(0)}, v_1^{(0)}, v_2^{(0)}, v_3^{(0)})$, eq.(2.26) reduces to

$$v_0^{(0)} = 1, \ v_1^{(0)} = t, \ \frac{\partial^2}{\partial t^2} v_2^{(0)} = K_{ttt}, \ \frac{\partial^2}{\partial t^2} v_3^{(0)} = -t K_{ttt}. \tag{2.27}$$

It is then straightforward to see that the prepotential is constructed as

$$F(t) = \frac{1}{2}(v_0^{(0)} v_3^{(0)} + v_1^{(0)} v_2^{(0)}) \tag{2.28}$$

such that $K_{ttt}(t) = \frac{\partial^3}{\partial t^3} F(t)$ [12]. From the transformation matrix M_k (2.16), we know the relation $v_{\gamma_j}^{(0)} = w_{\gamma_j}^{(0)}/r_{11} = w_{\gamma_j}^{(0)}/\bar{w}_0^{(0)}$. Therefore we can determine the prepotential from the solutions of the Picard-Fuchs equation.

3 Flat coordinates for a two-parameter model

In this section, we consider a two-parameter model. A mirror symmetry of two-parameter models have been intensively studied by Hosono et al. [9] and by Candelas et al. [10] [11]. In the following we concentrate on the model described by

$$M = \{x_i \in \mathbf{P}(1,1,2,2,2) | W = x_0^8 + x_1^8 + x_2^4 + x_3^4 + x_4^4\}. \tag{3.1}$$

This Calabi-Yau manifold has the Hodge numbers $h_{1,1} = 2$ and $h_{2,1} = 86$.

To obtain the mirror manifold of this model, one must consider the orbifoldization of M by dividing out a discrete symmetry $G = \mathbf{Z}_4^3$. The generators of G are given by

$$\mathbf{Z}_4 \ : \ (x_0, x_1, x_2, x_3, x_4) \to (x_0, e^{\frac{2\pi i}{4} 3} x_1, e^{\frac{2\pi i}{4}} x_2, x_3, x_4)$$

$$\mathbf{Z}_4 \ : \ (x_0, x_1, x_2, x_3, x_4) \to (x_0, e^{\frac{2\pi i}{4} 3} x_1, x_2, e^{\frac{2\pi i}{4}} x_3, x_4)$$

$$\mathbf{Z}_4 \ : \ (x_0, x_1, x_2, x_3, x_4) \to (x_0, e^{\frac{2\pi i}{4} 3} x_1, x_2, x_3, e^{\frac{2\pi i}{4}} x_4). \tag{3.2}$$

There are two marginal operators (degree eight monomials) which are invariant under this discrete symmetry. So, one can deform the defining relation in (3.1) with these operators. This deformation is regarded as the complex structure deformation of the mirror manifold of (3.1). Now the deformed polynomial is given by

$$W(\alpha, \beta) = W - 8\alpha x_0 x_1 x_2 x_3 x_4 - 2\beta x_0^4 x_1^4, \tag{3.3}$$

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where α and β are deformation parameters. Let $M_{\alpha,\beta}$ denote the manifold which is defined by the equation $W(\alpha,\beta) = 0$ on $\mathbf{P}(1,1,2,2,2)$, then one finds that the mirror manifold of $M_{\alpha,\beta}$ is realized as

$$\bar{M}_{\alpha,\beta} = M_{\alpha,\beta}/G. \tag{3.4}$$

This manifold actually has the Hodge numbers $h_{1,1} = 86$ and $h_{2,1} = 2$.

We introduce the deformed Jacobian ring like (2.4) to discuss the quantum cohomology ring of $\bar{M}_{\alpha,\beta}$ using the Gauss-Manin system. As the basis of the ring we choose

$$1,$$

$$x_0 x_1 x_2 x_3 x_4, \quad x_0^4 x_1^4,$$

$$x_0^2 x_1^2 x_2^2 x_3^2 x_4^2, \quad x_0^5 x_1^5 x_2 x_3 x_4,$$

$$x_0^6 x_1^6 x_2^2 x_3^2 x_4^2. \tag{3.5}$$

The period integrals are then defined by

$$w_{\gamma_j}^{(0)} = \int_{\gamma_j} \frac{1}{W(\alpha,\beta)} d\mu,$$

$$w_{\gamma_j}^{(1)} = -\int_{\gamma_j} \frac{x_0 x_1 x_2 x_3 x_4}{W^2(\alpha,\beta)} d\mu, \quad w_{\gamma_j}^{(2)} = -\int_{\gamma_j} \frac{x_0^4 x_1^4}{W^2(\alpha,\beta)} d\mu,$$

$$w_{\gamma_j}^{(3)} = 2 \int_{\gamma_j} \frac{x_0^2 x_1^2 x_2^2 x_3^2 x_4^2}{W^3(\alpha,\beta)} d\mu, \quad w_{\gamma_j}^{(4)} = 2 \int_{\gamma_j} \frac{x_0^5 x_1^5 x_2 x_3 x_4}{W^3(\alpha,\beta)} d\mu,$$

$$w_{\gamma_j}^{(5)} = -6 \int_{\gamma_j} \frac{x_0^6 x_1^6 x_2^2 x_3^2 x_4^2}{W^4(\alpha,\beta)} d\mu, \tag{3.6}$$

where $\gamma_0, \gamma_1, \cdots, \gamma_5$ are certain homology cycles in $H_3(\bar{M}_{\alpha,\beta}, \mathbf{Z})$. For the two-parameter model, these period integrals satisfy two sets of the first order differential equations

$$\frac{\partial}{\partial \alpha} w = G(\alpha,\beta) w, \quad \frac{\partial}{\partial \beta} w = H(\alpha,\beta) w, \tag{3.7}$$

where $w = {}^t(w^{(0)}, w^{(1)}, w^{(2)}, w^{(3)}, w^{(4)}, w^{(5)})$. Making use of the relations obtained from

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$\partial W(\alpha, \beta)/\partial x_i = \ldots$ and integrating by parts, we determine the matrices G and H

$$
G(\alpha, \beta) = \begin{pmatrix}
0 & -8 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -8 & 0 & 0 \\
0 & 0 & 0 & 0 & -8 & 0 \\
0 & 0 & -\alpha & 0 & 24\alpha^2 & -64\alpha^3 \\
0 & 0 & 0 & 0 & 0 & -8 \\
\frac{\alpha}{64\Delta} & \frac{-15\alpha^2}{8\Delta} & \frac{-\alpha(44\alpha^4+3\beta)}{4\Delta} & \frac{25\alpha^3}{\Delta} & \frac{2\alpha^2(148\alpha^4+13\beta)}{\Delta} & \frac{-128\alpha^3(8\alpha^4+\beta)}{\Delta}
\end{pmatrix},
$$

$$
H(\alpha, \beta) = \begin{pmatrix}
0 & 0 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & 0 \\
\frac{-1}{32Z} & \frac{3\alpha}{4Z} & \frac{3\beta}{2Z} & \frac{-2\alpha^2}{Z} & \frac{-4\alpha\beta}{Z} & 0 \\
0 & 0 & 0 & 0 & 0 & -2 \\
0 & \frac{-1}{8Z} & \frac{-\alpha^3}{4Z} & \frac{5\alpha}{4Z} & \frac{2(3\alpha^4+\beta)}{Z} & \frac{-4\alpha(4\alpha^4+\beta)}{Z} \\
\frac{\alpha^2(4\alpha^4+\beta)}{128Z\Delta} & \frac{-15\alpha^3(4\alpha^4+\beta)}{16Z\Delta} & H_{63} & H_{64} & H_{65} & H_{66}
\end{pmatrix}, \qquad (3.8)
$$

where

$$
\begin{aligned}
\Delta &= (8\alpha^4 + \beta)^2 - 1, \ Z = 1 - \beta^2, \\
H_{63} &= \frac{-\alpha^2(48\alpha^8 + 24\alpha^4\beta + \beta^2 + 2)}{8Z\Delta}, \\
H_{64} &= \frac{9 + 1024\alpha^8 + 256\alpha^4\beta - 9\beta^2}{32Z\Delta}, \\
H_{65} &= \frac{\alpha^3(7 + 144\alpha^8 + 88\alpha^4\beta + 6\beta^2)}{Z\Delta}, \\
H_{66} &= \frac{-52\alpha^4 - 768\alpha^{12} - 5\beta - 384\alpha^8\beta + 4\alpha^4\beta^2 + 5\beta^3}{2Z\Delta}.
\end{aligned} \qquad (3.9)
$$

These expressions have been obtained in [10].

From eqs.(3.7) and (3.8), one obtains five differential equations satisfied by the period $w^{(0)}$. However, some of them are not independent. By careful analysis, we find a set of independent equations. These equations are the Picard-Fuchs equations for $w^{(0)}$ whose explicit forms are

$$
w^{(0)(3,0)} - 32\alpha^3 w^{(0)(2,1)} - 96\alpha^2 w^{(0)(1,1)} - 32\alpha w^{(0)(0,1)} = 0,
$$

$$
16(\beta^2 - 1)w^{(0)(0,2)} + \alpha^2 w^{(0)(2,0)} + 8\alpha\beta w^{(0)(1,1)} + 3\alpha w^{(0)(1,0)} + 24\beta w^{(0)(0,1)} + w^{(0)} = 0,
$$

$$
(8\alpha^4 + \beta + 1)(8\alpha^4 + \beta - 1)w^{(0)(3.1)} + 128\alpha^3(8\alpha^4 + \beta)w^{(0)(2,1)}
$$

$$
+ 50\alpha^3 w^{(0)(2,0)} + 16\alpha^2(148\alpha^4 + 13\beta)w^{(0)(1,1)} + 30\alpha^2 w^{(0)(1,0)}
$$

$$
+ 16\alpha(44\alpha^4 + 3\beta)w^{(0)(0,1)} + 2\alpha w^{(0)} = 0, \qquad (3.10)
$$

where $w^{(0)(i,j)} = \frac{\partial^{i+j} w^{(0)}}{\partial \alpha^i \partial \beta^j}$ [10].

Now, to characterize the flat coordinates, we make a degree-preserving change of our basis of the Jacobian ring as

$$
\begin{array}{ccc}
\begin{array}{c}
1, \\
x_0 x_1 x_2 x_3 x_4, \quad x_0^4 x_1^4, \\
x_0^2 x_1^2 x_2^2 x_3^2 x_4^2, \quad x_0^5 x_1^5 x_2 x_3 x_4, \\
x_0^6 x_1^6 x_2^2 x_3^2 x_4^2,
\end{array}
& \rightarrow &
\begin{array}{c}
\mathcal{O}^{(0)}, \\
\mathcal{O}_t^{(1)}, \quad \mathcal{O}_s^{(1)}, \\
\mathcal{O}^{s(2)}, \quad \mathcal{O}^{t(2)}, \\
\mathcal{O}^{(3)},
\end{array}
\end{array}
\tag{3.11}
$$

with $\mathcal{O}^{(0)} = 1$. The superscript stands for the charge of $\mathcal{O}^{(i)}$ which is a polynomial of degree $8i$. Correspondingly we regard α and β as functions of t and s, i.e. $\alpha = \alpha(t, s)$ and $\beta = \beta(t, s)$. For this new basis, we can define the period integrals as given by (2.10). Let v denote the period vector in this coordinate, then v is related to w through

$$
w(\alpha, \beta) = M(\alpha, \beta) v(t, s), \tag{3.12}
$$

where

$$
M(\alpha, \beta) = \begin{pmatrix}
r_{11} & 0 & 0 & 0 & 0 & 0 \\
r_{21} & r_{22} & r_{23} & 0 & 0 & 0 \\
r_{31} & r_{32} & r_{33} & 0 & 0 & 0 \\
r_{41} & r_{42} & r_{43} & r_{44} & r_{45} & 0 \\
r_{51} & r_{52} & r_{53} & r_{54} & r_{55} & 0 \\
r_{61} & r_{62} & r_{63} & r_{64} & r_{65} & r_{66}
\end{pmatrix}. \tag{3.13}
$$

Notice that since there are two elements in charge one and two sectors, respectively, the matrix elements r_{23} and r_{45} are allowed to exist.

We constrain the basis $\{\mathcal{O}^{(0)}, \mathcal{O}_t^{(1)}, \mathcal{O}_s^{(1)}, \mathcal{O}^{s(2)}, \mathcal{O}^{t(2)}, \mathcal{O}^{(3)}\}$ to satisfy the flat coordinate conditions

$$
\begin{aligned}
&\text{i)} \quad \mathcal{O}_t^{(1)} = \frac{\partial}{\partial t}(r_{11} W), \quad \mathcal{O}_s^{(1)} = \frac{\partial}{\partial s}(r_{11} W), \\
&\text{ii)} \quad \mathcal{O}_i^{(1)} \mathcal{O}_j^{(1)} = K_{ijk} \mathcal{O}^{k(2)}, \quad (i, j, k = s, t), \\
&\text{iii)} \quad \mathcal{O}_i^{(1)} \mathcal{O}^{j(2)} = \delta_i^j \mathcal{O}^{(3)}, \quad (i, j = s, t), \\
&\text{iv)} \quad \mathcal{O} \mathcal{O}^{(3)} = 0, \quad (\mathcal{O} = \mathcal{O}^{(1)}, \mathcal{O}^{(2)}).
\end{aligned}
\tag{3.14}
$$

From these conditions, we derive two first order differential equations for v with respect

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to t and s. The result is

$$\frac{\partial}{\partial t}v = R_t v, \quad \frac{\partial}{\partial s}v = R_s v, \tag{3.15}$$

where

$$R_t = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{tts} & K_{ttt} & 0 \\ 0 & 0 & 0 & K_{tss} & K_{tst} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.16}$$

and

$$R_s = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{sts} & K_{stt} & 0 \\ 0 & 0 & 0 & K_{sss} & K_{sst} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.17}$$

Using (3.12), (3.15), R_t and R_s, the original Gauss-Manin system (3.7) is rewritten as

$$Av = 0, \quad Bv = 0, \tag{3.18}$$

where

$$A = \left(GM - \frac{\partial M}{\partial \alpha} - M\frac{\partial t}{\partial \alpha}R_t - M\frac{\partial s}{\partial \alpha}R_s \right) \tag{3.19}$$

and

$$B = \left(HM - \frac{\partial M}{\partial \beta} - M\frac{\partial t}{\partial \beta}R_t - M\frac{\partial s}{\partial \beta}R_s \right). \tag{3.20}$$

Since all the elements of v are linearly independent eq.(3.18) is equivalent to

$$A(r_{11}, t, s) = 0, \quad B(r_{11}, t, s) = 0 \tag{3.21}$$

which give rise to the differential equations obeyed by $r_{11}(\alpha, \beta)$, $t(\alpha, \beta)$ and $s(\alpha, \beta)$.

To write down them explicitly we first determine the transformation matrix $M(\alpha, \beta)$.

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From eqs.(3.18)$\sim$(3.20), we get

$$M(\alpha, \beta) =$$

$$\begin{pmatrix}
r_{11} & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{8}\frac{\partial}{\partial\alpha}r_{11} & -\frac{1}{8}r_{11}\frac{\partial}{\partial\alpha}t & -\frac{1}{8}r_{11}\frac{\partial}{\partial\alpha}s & 0 & 0 & 0 \\
-\frac{1}{2}\frac{\partial}{\partial\beta}r_{11} & -\frac{1}{2}r_{11}\frac{\partial}{\partial\beta}t & -\frac{1}{2}r_{11}\frac{\partial}{\partial\beta}s & 0 & 0 & 0 \\
\frac{1}{64}\frac{\partial^2}{\partial\alpha^2}r_{11} & \frac{1}{64}r_{11}\frac{\partial^2}{\partial\alpha^2}t + \frac{1}{32}\frac{\partial}{\partial\alpha}r_{11}\frac{\partial}{\partial\alpha}t & \frac{1}{64}r_{11}\frac{\partial^2}{\partial\alpha^2}s + \frac{1}{32}\frac{\partial}{\partial\alpha}r_{11}\frac{\partial}{\partial\alpha}s & r_{44} & r_{45} & 0 \\
\frac{1}{16}\frac{\partial^2}{\partial\alpha\partial\beta}r_{11} & r_{52} & r_{53} & r_{54} & r_{55} & 0 \\
-\frac{1}{128}\frac{\partial^3}{\partial\alpha^2\partial\beta}r_{11} & r_{62} & r_{63} & r_{64} & r_{65} & r_{66}
\end{pmatrix},$$
(3.22)

where the explicit forms of r_{ij} are given in appendix (see eqs.(A.2)$\sim$(A.12)). Substituting this transformation matrix into (3.19) and (3.20), we see $K_{tts} = K_{tst} = K_{stt}$ and $K_{sst} = K_{sts} = K_{tss}$. If we define

$$K_{\alpha\alpha\alpha} = -\frac{4096\alpha^3}{\Delta}, \quad K_{\alpha\alpha\beta} = -\frac{128}{\Delta},$$

$$K_{\alpha\beta\beta} = -\frac{64\alpha(\beta + 4\alpha^4)}{(1 - \beta^2)\Delta}, \quad K_{\beta\beta\beta} = -\frac{8\alpha^2(1 + 3\beta^2 + 16\alpha^4\beta)}{(1 - \beta^2)^2\Delta}, \tag{3.23}$$

where $\Delta = (8\alpha^4 + \beta)^2 - 1$, we find

$$K_{ttt} = \frac{C}{r_{11}^2}\left[K_{\alpha\alpha\alpha}\left(\frac{\partial\alpha}{\partial t}\right)^3 + 3K_{\alpha\alpha\beta}\left(\frac{\partial\alpha}{\partial t}\right)^2\frac{\partial\beta}{\partial t} + 3K_{\alpha\beta\beta}\frac{\partial\alpha}{\partial t}\left(\frac{\partial\beta}{\partial t}\right)^2 + K_{\beta\beta\beta}\left(\frac{\partial\beta}{\partial t}\right)^3\right],$$

$$\begin{aligned}
K_{tts} &= \frac{C}{r_{11}^2}\left[K_{\alpha\alpha\alpha}\left(\frac{\partial\alpha}{\partial t}\right)^2\frac{\partial\alpha}{\partial s} + K_{\alpha\alpha\beta}\left(\left(\frac{\partial\alpha}{\partial t}\right)^2\frac{\partial\beta}{\partial s} + 2\frac{\partial\alpha}{\partial t}\frac{\partial\alpha}{\partial s}\frac{\partial\beta}{\partial t}\right)\right. \\
&\quad\left. + K_{\alpha\beta\beta}\left(\frac{\partial\alpha}{\partial s}\left(\frac{\partial\beta}{\partial t}\right)^2 + 2\frac{\partial\alpha}{\partial t}\frac{\partial\beta}{\partial s}\frac{\partial\beta}{\partial t}\right) + K_{\beta\beta\beta}\left(\frac{\partial\beta}{\partial t}\right)^2\frac{\partial\beta}{\partial s}\right], \tag{3.24}
\end{aligned}$$

where C is an arbitrary constant and we have used

$$\begin{pmatrix} \frac{\partial\alpha}{\partial t} & \frac{\partial\beta}{\partial t} \\ \frac{\partial\alpha}{\partial s} & \frac{\partial\beta}{\partial s} \end{pmatrix} = \begin{pmatrix} \frac{\partial t}{\partial\alpha} & \frac{\partial s}{\partial\alpha} \\ \frac{\partial t}{\partial\beta} & \frac{\partial s}{\partial\beta} \end{pmatrix}^{-1} = \frac{1}{\frac{\partial t}{\partial\alpha}\frac{\partial s}{\partial\beta} - \frac{\partial t}{\partial\beta}\frac{\partial s}{\partial\alpha}}\begin{pmatrix} \frac{\partial s}{\partial\beta} & -\frac{\partial s}{\partial\alpha} \\ -\frac{\partial t}{\partial\beta} & \frac{\partial t}{\partial\alpha} \end{pmatrix} \tag{3.25}$$

to convert $\frac{\partial t}{\partial\alpha}, \frac{\partial t}{\partial\beta}, \frac{\partial s}{\partial\alpha}, \frac{\partial s}{\partial\beta}$ into $\frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial s}, \frac{\partial\beta}{\partial t}, \frac{\partial\beta}{\partial s}$. K_{sss} and K_{tss} are obtained from K_{ttt} and K_{tts} by interchanging t and s. The form of (3.24) is in agreement with [10]. Substituting

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(3.22) and K_{ijk} into (3.19) and (3.20), we find the matrices A and B as follows

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ A_{41}(r_{11}) & A_{42}(r_{11},t) & A_{43}(r_{11},s) & A_{44}(r_{11},t,s) & A_{45}(r_{11},t,s) & 0 \\ 0 & 0 & 0 & A_{54}(r_{11},t,s) & A_{55}(r_{11},t,s) & 0 \\ A_{61}(r_{11}) & A_{62}(r_{11},t) & A_{63}(r_{11},s) & A_{64}(r_{11},t,s) & A_{65}(r_{11},t,s) & 0 \end{pmatrix} \tag{3.26}$$

and

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ B_{31}(r_{11}) & B_{32}(r_{11},t) & B_{33}(r_{11},s) & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{44}(r_{11},t,s) & B_{45}(r_{11},t,s) & 0 \\ B_{51}(r_{11}) & B_{52}(r_{11},t) & B_{53}(r_{11},s) & B_{54}(r_{11},t,s) & B_{55}(r_{11},t,s) & 0 \\ B_{61}(r_{11}) & B_{62}(r_{11},t) & B_{63}(r_{11},s) & B_{64}(r_{11},t,s) & B_{65}(r_{11},t,s) & 0 \end{pmatrix}. \tag{3.27}$$

Now, $A_{ij} = 0$ and $B_{ij} = 0$ for all i, j are non-trivial differential equations which must be satisfied by t, s and r_{11}. Explicit forms of A_{ij} and B_{ij} are given in appendix.

Here we explain some detailed properties of A_{ij} and B_{ij}.

(i) The relations for the elements of the first column: The equations $A_{41}(r_{11}) = 0$, $A_{61}(r_{11}) = 0$ and $B_{31}(r_{11}) = 0$ are proportional to the Picard-Fuchs equations (3.10) (see eqs.(A.13)$\sim$(A.15) in appendix). So, r_{11} is a solution of the Picard-Fuchs equations. We also notice that $B_{51}(r_{11})$ and $B_{61}(r_{11})$ are not independent. They are expressed in terms of $A_{41}(r_{11})$, $A_{61}(r_{11})$, $B_{31}(r_{11})$ and their derivatives.

(ii) The relations for the elements of the second and third columns: We have

$$A_{i2}(r_{11}, t) = A_{i3}(r_{11}, t) \tag{3.28}$$

for $i = 4, 6$ and

$$B_{i2}(r_{11}, t) = B_{i3}(r_{11}, t) \tag{3.29}$$

for $i = 3, 5$ and 6. In addition, $B_{52}(r_{11}, t)$ and $B_{62}(r_{11}, t)$ are not independent. They are expressed in terms of $A_{42}(r_{11}, t)$, $A_{62}(r_{11}, t)$, $B_{32}(r_{11}, t)$ and their derivatives. Moreover, let $\bar{w}_j^{(0)}$ ($0 \le j \le 5$) denote the solutions of the Picard-Fuchs equations (3.10), then we

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find that the solutions of $A_{42}(r_{11}, t) = 0$, $A_{62}(r_{11}, t) = 0$ and $B_{32}(r_{11}, t) = 0$ are expressed by

$$t = \sum_{j=0}^{5} \frac{c_j \bar{w}_j^{(0)}}{r_{11}}, \tag{3.30}$$

where c_j's are arbitrary constants. Similarly, for s, we obtain

$$s = \sum_{j=0}^{5} \frac{a_j \bar{w}_j^{(0)}}{r_{11}} \tag{3.31}$$

with a_j being arbitrary constants.

(iii) The relations for the elements of the fourth and fifth columns: We have

$$A_{i4}(r_{11}, t, s) = A_{i5}(r_{11}, s, t), \quad B_{i4}(r_{11}, t, s) = B_{i5}(r_{11}, s, t) \tag{3.32}$$

for $i = 4, 5$ and 6. We also find

$$4 \frac{\partial t}{\partial \beta} A_{44}(r_{11}, t, s) = \frac{\partial t}{\partial \alpha} A_{54}(r_{11}, t, s). \tag{3.33}$$

Hence, among the elements of the fourth and fifth columns, we are left with five elements, say A_{44}, A_{64}, B_{44}, B_{54} and B_{64}, which seem to be independent. Are these five elements really independent? As long as we have manipulated these equations explicitly we feel that there still exist some relations which can be used to reduce the number of independent equations. Unfortunately the equations we have to deal with are so complicated that we have not yet succeeded in finding them.

To summarize, analyzing the Gauss-Manin system in the flat coordinates we have obtained the equations for r_{11}

$$A_{41}(r_{11}) = 0, \quad A_{61}(r_{11}) = 0, \quad B_{31}(r_{11}) = 0, \tag{3.34}$$

which are equivalent to the Picard-Fuchs equations (3.10). This is the property (i). The property (ii) is that the flat coordinates t and s take the form (3.30) and (3.31), respectively. In order to fix c_j and a_j we have to solve

$$A_{44}(r_{11}, t, s) = 0, \quad A_{64}(r_{11}, t, s) = 0, \quad B_{44}(r_{11}, t, s) = 0,$$

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$$B_{54}(r_{11}, t, s) = 0, \quad B_{64}(r_{11}, t, s) = 0 \tag{3.35}$$

according to the property (iii), though we are not sure if these five equations are all independent.

We now wish to solve our differential equations (3.34) and (3.35). First of all, from the explicit calculations of period integral and the analytic continuation, one can find a solution to the Picard-Fuchs equations. The result is [10] [23]

$$\bar{w}_0^{(0)}(\alpha, \beta) = -\frac{1}{4\alpha} \sum_{n=1}^{\infty} \frac{\Gamma(\frac{n}{4})(-8\alpha)^n}{\Gamma(n)\Gamma^3(1 - \frac{n}{4})} U_{-\frac{n}{4}}(\beta), \quad |\alpha| \ll 1 \tag{3.36}$$

where

$$U_\nu(\beta) = \frac{e^{i\pi\nu/2}}{2\Gamma(-\nu)} \sum_{m=0}^{\infty} \frac{e^{i\pi m/2}\Gamma(\frac{m-\nu}{2})}{m!\Gamma(1 - \frac{m-\nu}{2})} (2\beta)^m, \quad |\beta| < 1. \tag{3.37}$$

Then the other solutions are given by

$$\bar{w}_j^{(0)}(\alpha, \beta) = \bar{w}_0^{(0)}(e^{\frac{i\pi}{4}j}\alpha, (-1)^j\beta) \tag{3.38}$$

for $j = 1, 2, \cdots, 5$.

Let us next turn to (3.35). We notice that the forms of t and s in (3.30), (3.31) look like the mirror map for the two-parameter model if we take $r_{11} \propto \bar{w}_0^{(0)}$. In fact the mirror map found by Candelas et al. is given by [10]

$$t(\alpha, \beta) = -\frac{1}{2} + \frac{1}{4\bar{w}_0^{(0)}}[(\bar{w}_0^{(0)} + 2\bar{w}_2^{(0)} + \bar{w}_4^{(0)}) + (3\bar{w}_1^{(0)} + 2\bar{w}_3^{(0)} + \bar{w}_5^{(0)})], \tag{3.39}$$

$$s(\alpha, \beta) = -\frac{1}{2} + \frac{1}{4\bar{w}_0^{(0)}}[(\bar{w}_0^{(0)} + 2\bar{w}_2^{(0)} + \bar{w}_4^{(0)}) - (3\bar{w}_1^{(0)} + 2\bar{w}_3^{(0)} + \bar{w}_5^{(0)})]. \tag{3.40}$$

Thus we assume

$$r_{11} = \bar{w}_0^{(0)}(\alpha, \beta), \quad t = t(\alpha, \beta), \quad s = s(\alpha, \beta) \tag{3.41}$$

and check if these satisfy (3.35). Actually, after tedious calculations, we have confirmed that (3.35) is satisfied by (3.41). This is an extremely non-trivial result. Thus, in the two-parameter case, we have also derived the differential equations which govern the periods and the mirror maps.

It should be remarked that $t = t(\alpha, \beta)$ and $s = s(\alpha, \beta)$ are not unique solutions of (3.35). In fact $\tilde{t} = a\, t(\alpha, \beta) + b\, s(\alpha, \beta)$ and $\tilde{s} = c\, t(\alpha, \beta) + d\, s(\alpha, \beta)$ are also solutions, where a, b, c and d are arbitrary constants with $ad \neq bc$. We discuss this point at the end of appendix.

Finally we derive the prepotential from the solutions of the Gauss-Manin system in the flat coordinates (3.15). It is not difficult to see that eq.(3.15) is equivalent to

$$
\begin{aligned}
\frac{\partial^2}{\partial x \partial y}\left\{\frac{K_{sss}}{K_{ttt}K_{sss} - K_{tts}K_{sst}}\frac{\partial^2}{\partial t^2} - \frac{K_{tts}}{K_{ttt}K_{sss} - K_{tts}K_{sst}}\frac{\partial^2}{\partial s^2}\right\} v^{(0)} &= 0, \\
\frac{\partial^2}{\partial x \partial y}\left\{\frac{K_{tss}}{K_{ttt}K_{tss} - K_{tts}^2}\frac{\partial^2}{\partial t^2} - \frac{K_{tts}}{K_{ttt}K_{tss} - K_{tts}^2}\frac{\partial^2}{\partial s \partial t}\right\} v^{(0)} &= 0, \\
\frac{\partial^2}{\partial x \partial y}\left\{\frac{K_{tts}}{K_{ttt}K_{tss} - K_{tts}^2}\frac{\partial^2}{\partial t^2} - \frac{K_{ttt}}{K_{ttt}K_{tss} - K_{tts}^2}\frac{\partial^2}{\partial s \partial t}\right\} v^{(0)} &= 0 \qquad (3.42)
\end{aligned}
$$

with $x, y = s, t$, and those obtained from (3.42) by interchanging t and s. For $v^{(0)} = {}^t(v_0^{(0)}, v_1^{(0)}, v_2^{(0)}, v_3^{(0)}, v_4^{(0)}, v_5^{(0)})$, eq.(3.42) reduces to

$$
v_0^{(0)} = 1, \quad v_1^{(0)} = t, \quad v_2^{(0)} = s,
$$

$$
v_3^{(0)}: \quad \frac{\partial^2}{\partial t^2}v_3^{(0)} = K_{stt}, \quad \frac{\partial^2}{\partial s \partial t}v_3^{(0)} = K_{tss}, \quad \frac{\partial^2}{\partial s^2}v_3^{(0)} = K_{sss},
$$

$$
v_4^{(0)}: \quad \frac{\partial^2}{\partial t^2}v_4^{(0)} = K_{ttt}, \quad \frac{\partial^2}{\partial s \partial t}v_4^{(0)} = K_{tts}, \quad \frac{\partial^2}{\partial s^2}v_4^{(0)} = K_{tss},
$$

$$
v_5^{(0)}: \qquad\qquad \frac{\partial^2}{\partial t^2}v_5^{(0)} = -(tK_{ttt} + sK_{stt}),
$$

$$
\frac{\partial^2}{\partial s \partial t}v_5^{(0)} = -(tK_{tts} + sK_{sst}),
$$

$$
\frac{\partial^2}{\partial s^2}v_5^{(0)} = -(tK_{tss} + sK_{sss}). \qquad (3.43)
$$

Then it is easy to show that the prepotential is given by

$$
F(t, s) = \frac{1}{2}(v_0^{(0)}v_5^{(0)} + v_1^{(0)}v_4^{(0)} + v_2^{(0)}v_3^{(0)}). \qquad (3.44)
$$

Of course, this is obtained so as to have $K_{xyz} = \frac{\partial^3}{\partial x \partial y \partial z}F(t, s)$, where $x, y, z = s, t$. From the transformation matrix (3.22), we know the relation $v_{\gamma_j}^{(0)} = w_{\gamma_j}^{(0)}/r_{11} = w_{\gamma_j}^{(0)}/\bar{w}_0^{(0)}$, from which we can evaluate the prepotential.

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4 Conclusions

In this article we have studied mirror symmetry of Calabi-Yau manifolds by formulating the Gauss-Manin system in the flat coordinates. Following the work by Hosono-Lian [12] on the quintic hypersurface we have shown for various examples of Calabi-Yau manifolds that the Gauss-Manin system yields the differential equations obeyed by the mirror map in addition to the ordinary Picard-Fuchs equations.

It is very interesting that the mirror map is governed by differential equations such as (2.17) for one-parameter models and (3.35) for a two-parameter model. Concerning the two-parameter model we expect that the system of equations (3.35) is still redundant and it may reduce to more fundamental one.

When checking (2.17) and (3.35) we have assumed the form of the mirror map. It is desirable to find a way to solve (2.17) and (3.35) so that one can construct the mirror map directly as a solution to the differential equations. Furthermore it will be important to understand the deeper structure and the physical meaning of the differential equations for the mirror map in view of duality symmetry.

Acknowledgements

Appendix

In this appendix, we present the matrix elements of $M(\alpha, \beta)$, A and B for the two-parameter model in section 3. To simplify the expressions, we introduce the following notations

$$Z = 1 - \beta^2, \ \Delta = 64\alpha^8 + 16\alpha^4\beta + \beta^2 - 1, \ D = \frac{\partial t}{\partial \alpha}\frac{\partial s}{\partial \beta} - \frac{\partial t}{\partial \beta}\frac{\partial s}{\partial \alpha}. \tag{A.1}$$

First, we give the elements of $M(\alpha, \beta)$. They are

$$r_{44} = \frac{2C}{\Delta\, D\, r_{11}}\left(32\alpha^3\frac{\partial t}{\partial \beta} - \frac{\partial t}{\partial \alpha}\right), \tag{A.2}$$

$$r_{45} = \frac{-2C}{\Delta\, D\, r_{11}}\left(32\alpha^3\frac{\partial s}{\partial \beta} - \frac{\partial s}{\partial \alpha}\right), \tag{A.3}$$

$$r_{52} = \frac{1}{16}r_{11}\frac{\partial^2 t}{\partial\alpha\partial\beta} + \frac{1}{16}\frac{\partial r_{11}}{\partial \alpha}\frac{\partial t}{\partial \beta} + \frac{1}{16}\frac{\partial r_{11}}{\partial \beta}\frac{\partial t}{\partial \alpha}, \tag{A.4}$$

$$r_{53} = \frac{1}{16}r_{11}\frac{\partial^2 s}{\partial\alpha\partial\beta} + \frac{1}{16}\frac{\partial r_{11}}{\partial \alpha}\frac{\partial s}{\partial \beta} + \frac{1}{16}\frac{\partial r_{11}}{\partial \beta}\frac{\partial s}{\partial \alpha}, \tag{A.5}$$

$$r_{54} = \frac{-4C}{Z\, \Delta\, D\, r_{11}}\left(4\alpha^5\frac{\partial t}{\partial \alpha} + \alpha\beta\frac{\partial t}{\partial \alpha} - 2\frac{\partial t}{\partial \beta} + 2\beta^2\frac{\partial t}{\partial \beta}\right), \tag{A.6}$$

$$r_{55} = \frac{4C}{Z\, \Delta\, D\, r_{11}}\left(4\alpha^5\frac{\partial s}{\partial \alpha} + \alpha\beta\frac{\partial s}{\partial \alpha} - 2\frac{\partial s}{\partial \beta} + 2\beta^2\frac{\partial s}{\partial \beta}\right), \tag{A.7}$$

$$\begin{aligned}
r_{62} &= -\frac{1}{128}r_{11}\frac{\partial^3 t}{\partial\alpha^2\partial\beta} - \frac{1}{64}\frac{\partial r_{11}}{\partial \alpha}\frac{\partial^2 t}{\partial\alpha\partial\beta} - \frac{1}{128}\frac{\partial r_{11}}{\partial \beta}\frac{\partial^2 t}{\partial\alpha^2} \\
&\quad - \frac{1}{128}\frac{\partial^2 r_{11}}{\partial\alpha^2}\frac{\partial t}{\partial \beta} - \frac{1}{64}\frac{\partial^2 r_{11}}{\partial\alpha\partial\beta}\frac{\partial t}{\partial \alpha},
\end{aligned} \tag{A.8}$$

$$\begin{aligned}
r_{63} &= -\frac{1}{128}r_{11}\frac{\partial^3 s}{\partial\alpha^2\partial\beta} - \frac{1}{64}\frac{\partial r_{11}}{\partial \alpha}\frac{\partial^2 s}{\partial\alpha\partial\beta} - \frac{1}{128}\frac{\partial r_{11}}{\partial \beta}\frac{\partial^2 s}{\partial\alpha^2} \\
&\quad - \frac{1}{128}\frac{\partial^2 r_{11}}{\partial\alpha^2}\frac{\partial s}{\partial \beta} - \frac{1}{64}\frac{\partial^2 r_{11}}{\partial\alpha\partial\beta}\frac{\partial s}{\partial \alpha},
\end{aligned} \tag{A.9}$$

$$\begin{aligned}
r_{64} &= \frac{C(-768\alpha^{12} - 384\alpha^8\beta - 20\alpha^4 - 28\alpha^4\beta^2 + \beta^3 - \beta)}{2\, Z\, \Delta^2\, D\, r_{11}}\frac{\partial t}{\partial \alpha} \\
&\quad + \frac{64C\alpha^3(8\alpha^4 + \beta)}{\Delta^2\, D\, r_{11}}\frac{\partial t}{\partial \beta} + \frac{C}{\Delta\, D\, r_{11}^2}\left(\frac{\partial t}{\partial \alpha}\frac{\partial r_{11}}{\partial \beta} - \frac{\partial t}{\partial \beta}\frac{\partial r_{11}}{\partial \alpha}\right),
\end{aligned} \tag{A.10}$$

$$r_{65} = -\frac{C(-768\alpha^{12} - 384\alpha^8\beta - 20\alpha^4 - 28\alpha^4\beta^2 + \beta^3 - \beta)}{2\,Z\,\Delta^2\,D\,r_{11}}\frac{\partial s}{\partial\alpha}$$
$$-\frac{64C\alpha^3(8\alpha^4 + \beta)}{\Delta^2\,D\,r_{11}}\frac{\partial s}{\partial\beta} - \frac{C}{\Delta\,D\,r_{11}^2}\left(\frac{\partial s}{\partial\alpha}\frac{\partial r_{11}}{\partial\beta} - \frac{\partial s}{\partial\beta}\frac{\partial r_{11}}{\partial\alpha}\right) \qquad \text{(A.11)}$$

and

$$r_{66} = \frac{C}{\Delta\,r_{11}}. \qquad \text{(A.12)}$$

Next, we give some elements of A and B. For the elements of the first column of A and B, we have

$$A_{41}(r_{11}) = \frac{1}{2}\alpha\frac{\partial r_{11}}{\partial\beta} + \frac{3}{2}\alpha^2\frac{\partial^2 r_{11}}{\partial\alpha\partial\beta} + \frac{1}{2}\alpha^3\frac{\partial^3 r_{11}}{\partial\alpha^2\partial\beta} - \frac{1}{64}\frac{\partial^3 r_{11}}{\partial\alpha^3}, \qquad \text{(A.13)}$$

$$A_{61}(r_{11}) = \frac{\alpha}{64\,\Delta}r_{11} + \frac{15\alpha^2}{64\,\Delta}\frac{\partial r_{11}}{\partial\alpha} + \frac{\alpha(44\alpha^4 + 3\beta)}{8\,\Delta}\frac{\partial r_{11}}{\partial\beta} + \frac{25\alpha^3}{64\,\Delta}\frac{\partial^2 r_{11}}{\partial\alpha^2}$$
$$+ \frac{\alpha^2(148\alpha^4 + 13\beta)}{8\,\Delta}\frac{\partial^2 r_{11}}{\partial\alpha\partial\beta} + \frac{\alpha^3(8\alpha^4 + \beta)}{\Delta}\frac{\partial^3 r_{11}}{\partial\alpha^2\partial\beta}$$
$$+ \frac{1}{128}\frac{\partial^4 r_{11}}{\partial\alpha^3\partial\beta} \qquad \text{(A.14)}$$

and

$$B_{31}(r_{11}) = -\frac{1}{32\,Z}r_{11} - \frac{3\alpha}{32\,Z}\frac{\partial r_{11}}{\partial\alpha} - \frac{3\beta}{4\,Z}\frac{\partial r_{11}}{\partial\beta} - \frac{\alpha^2}{32\,Z}\frac{\partial^2 r_{11}}{\partial\alpha^2}$$
$$- \frac{\alpha\beta}{4\,Z}\frac{\partial^2 r_{11}}{\partial\alpha\partial\beta} + \frac{1}{2}\frac{\partial^2 r_{11}}{\partial\beta^2}. \qquad \text{(A.15)}$$

These are proportional to the Picard-Fuchs equations (3.10). B_{51} and B_{61} are given by

$$B_{51}(r_{11}) = \frac{1}{Z}\left(\frac{\alpha^2}{4}A_{41}(r_{11}) - \frac{Z}{8}\frac{\partial}{\partial\alpha}B_{31}(r_{11})\right) \qquad \text{(A.16)}$$

and

$$B_{61}(r_{11}) = \frac{1}{Z}\left\{\left(2\alpha^5 + \frac{\alpha\beta}{2}\right)A_{61}(r_{11}) - \frac{7\alpha}{32}A_{41}(r_{11})\right.$$
$$\left. - \frac{\alpha^2}{32}\frac{\partial}{\partial\alpha}A_{41}(r_{11}) + \frac{Z}{64}\frac{\partial^2}{\partial\alpha^2}B_{31}(r_{11})\right\}. \qquad \text{(A.17)}$$

For the elements of the second and third columns of A and B, we have

$$A_{42}(r_{11}, t) = \left(\frac{\alpha}{2}\frac{\partial t}{\partial\beta} + \frac{3\alpha^2}{2}\frac{\partial^2 t}{\partial\alpha\partial\beta} + \frac{\alpha^3}{2}\frac{\partial^3 t}{\partial\alpha^2\partial\beta} - \frac{1}{64}\frac{\partial^3 t}{\partial\alpha^3}\right)r_{11}$$

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$$+\left(\frac{3\alpha^2}{2}\frac{\partial t}{\partial\beta}+\alpha^3\frac{\partial^2 t}{\partial\alpha\partial\beta}-\frac{3}{64}\frac{\partial^2 t}{\partial\alpha^2}\right)\frac{\partial r_{11}}{\partial\alpha}$$
$$+\left(\frac{3\alpha^2}{2}\frac{\partial t}{\partial\alpha}+\frac{\alpha^3}{2}\frac{\partial^2 t}{\partial\alpha^2}\right)\frac{\partial r_{11}}{\partial\beta}+\left(\frac{\alpha^3}{2}\frac{\partial t}{\partial\beta}-\frac{3}{64}\frac{\partial t}{\partial\alpha}\right)\frac{\partial^2 r_{11}}{\partial\alpha^2}$$
$$+\alpha^3\frac{\partial t}{\partial\alpha}\frac{\partial^2 r_{11}}{\partial\alpha\partial\beta}, \tag{A.18}$$

$$\begin{aligned}
A_{62}(r_{11},t) =\ & \frac{15\,\alpha^2}{64\,\Delta}r_{11}\frac{\partial t}{\partial\alpha}+\frac{\alpha\,(44\alpha^5+3\beta)}{8\,\Delta}r_{11}\frac{\partial t}{\partial\beta}+\frac{25\,\alpha^3}{64\,\Delta}r_{11}\frac{\partial^2 t}{\partial\alpha^2}\\
&+\frac{\alpha^2\,(148\alpha^4+13\beta)}{8\,\Delta}r_{11}\frac{\partial^2 t}{\partial\alpha\partial\beta}+\frac{\alpha^3\,(8\alpha^4+\beta)}{\Delta}r_{11}\frac{\partial^3 t}{\partial\alpha^2\partial\beta}\\
&+\frac{1}{128}r_{11}\frac{\partial^4 t}{\partial\alpha^3\partial\beta}+\frac{25\,\alpha^3}{32\,\Delta}\frac{\partial r_{11}}{\partial\alpha}\frac{\partial t}{\partial\alpha}\\
&+\frac{\alpha^2\,(148\alpha^4+13\beta)}{8\,\Delta}\frac{\partial r_{11}}{\partial\alpha}\frac{\partial t}{\partial\beta}+\frac{2\,\alpha^3\,(8\alpha^4+\beta)}{\Delta}\frac{\partial r_{11}}{\partial\alpha}\frac{\partial^2 t}{\partial\alpha\partial\beta}\\
&+\frac{3}{128}\frac{\partial r_{11}}{\partial\alpha}\frac{\partial^3 t}{\partial\alpha^2\partial\beta}+\frac{\alpha^2\,(148\alpha^4+13\beta)}{8\,\Delta}\frac{\partial r_{11}}{\partial\beta}\frac{\partial t}{\partial\alpha}\\
&+\frac{\alpha^3\,(8\alpha^4+\beta)}{\Delta}\frac{\partial r_{11}}{\partial\beta}\frac{\partial^2 t}{\partial\alpha^2}+\frac{1}{128}\frac{\partial r_{11}}{\partial\beta}\frac{\partial^3 t}{\partial\alpha^3}\\
&+\frac{\alpha^3\,(8\alpha^4+\beta)}{\Delta}\frac{\partial^2 r_{11}}{\partial\alpha^2}\frac{\partial t}{\partial\beta}+\frac{3}{128}\frac{\partial^2 r_{11}}{\partial\alpha^2}\frac{\partial^2 t}{\partial\alpha\partial\beta}\\
&+\frac{2\,\alpha^3\,(8\alpha^4+\beta)}{\Delta}\frac{\partial^2 r_{11}}{\partial\alpha\partial\beta}\frac{\partial t}{\partial\alpha}+\frac{3}{128}\frac{\partial^2 r_{11}}{\partial\alpha\partial\beta}\frac{\partial^2 t}{\partial\alpha^2}\\
&+\frac{1}{128}\frac{\partial^3 r_{11}}{\partial\alpha^3}\frac{\partial t}{\partial\beta}+\frac{3}{128}\frac{\partial^3 r_{11}}{\partial\alpha^2\partial\beta}\frac{\partial t}{\partial\alpha} \tag{A.19}
\end{aligned}$$

and

$$\begin{aligned}
B_{32}(r_{11},t) =\ & -\frac{3\,\alpha}{32\,Z}r_{11}\frac{\partial t}{\partial\alpha}-\frac{3\,\beta}{4\,Z}r_{11}\frac{\partial t}{\partial\beta}-\frac{\alpha^2}{32\,Z}r_{11}\frac{\partial^2 t}{\partial\alpha^2}-\frac{\alpha\,\beta}{4\,Z}r_{11}\frac{\partial^2 t}{\partial\alpha\partial\beta}\\
&+\frac{1}{2}r_{11}\frac{\partial^2 t}{\partial\beta^2}-\frac{\alpha^2}{16\,Z}\frac{\partial r_{11}}{\partial\alpha}\frac{\partial t}{\partial\alpha}-\frac{\alpha\,\beta}{4\,Z}\frac{\partial r_{11}}{\partial\alpha}\frac{\partial t}{\partial\beta}\\
&-\frac{\alpha\,\beta}{4\,Z}\frac{\partial r_{11}}{\partial\beta}\frac{\partial t}{\partial\alpha}+\frac{\partial r_{11}}{\partial\beta}\frac{\partial t}{\partial\beta}. \tag{A.20}
\end{aligned}$$

Since there exist the relations (3.28) and (3.29), we only consider the elements of the second column. $B_{52}(r_{11},t)$ and $B_{62}(r_{11},t)$ are given by the equations which are similar to (A.16) and (A.17).

For the elements of the fourth and fifth columns of A and B, we have

$$A_{44}(r_{11},t,s) = \frac{C}{D^2\,\Delta\,r_{11}}\frac{\partial t}{\partial\alpha}\left(\frac{\alpha\,(4\alpha^4+\beta)}{Z}\left(\frac{\partial t}{\partial\alpha}\frac{\partial^2 s}{\partial\alpha^2}-\frac{\partial^2 t}{\partial\alpha^2}\frac{\partial s}{\partial\alpha}\right)\right.$$

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$$+64\ \alpha^3 \left(\frac{\partial t}{\partial \beta} \frac{\partial^2 s}{\partial \alpha \partial \beta} - \frac{\partial^2 t}{\partial \alpha \partial \beta} \frac{\partial s}{\partial \beta} \right)$$

$$+2 \left(\frac{\partial^2 t}{\partial \alpha \partial \beta} \frac{\partial s}{\partial \alpha} - \frac{\partial t}{\partial \alpha} \frac{\partial^2 s}{\partial \alpha \partial \beta} \right)$$

$$+2 \left(\frac{\partial^2 t}{\partial \alpha^2} \frac{\partial s}{\partial \beta} - \frac{\partial t}{\partial \beta} \frac{\partial^2 s}{\partial \alpha^2} \right)$$

$$+\frac{2\ D}{r_{11}} \frac{\partial r_{11}}{\partial \alpha} - \frac{64\ \alpha^3\ D}{r_{11}} \frac{\partial r_{11}}{\partial \beta} \right), \qquad \text{(A.21)}$$

$$
\begin{aligned}
B_{44}(r_{11}, t, s) \ &= \ \frac{C}{D^2\ \Delta\ r_{11}} \left(2 \frac{\partial t}{\partial \alpha} \left(\frac{\partial^2 t}{\partial \beta^2} \frac{\partial s}{\partial \alpha} - \frac{\partial t}{\partial \alpha} \frac{\partial^2 s}{\partial \beta^2} \right) + \frac{3\ (4\alpha^4 + \beta)\ D}{Z} \frac{\partial t}{\partial \alpha} \right. \\
&\quad + 2 \frac{\partial t}{\partial \beta} \left(\frac{\partial t}{\partial \beta} \frac{\partial^2 s}{\partial \alpha^2} - \frac{\partial^2 t}{\partial \alpha^2} \frac{\partial s}{\partial \beta} \right) \\
&\quad + \frac{\alpha^2\ (16\alpha^4 \beta + 3\beta^2 + 1)}{8\ Z^2} \frac{\partial t}{\partial \alpha} \left(\frac{\partial t}{\partial \alpha} \frac{\partial^2 s}{\partial \alpha^2} - \frac{\partial^2 t}{\partial \alpha^2} \frac{\partial s}{\partial \alpha} \right) \\
&\quad + \frac{\alpha\ (4\alpha^4 + \beta)}{Z} \frac{\partial t}{\partial \beta} \left(\frac{\partial^2 t}{\partial \alpha^2} \frac{\partial s}{\partial \alpha} - \frac{\partial t}{\partial \alpha} \frac{\partial^2 s}{\partial \alpha^2} \right) \\
&\quad + 64\ \alpha^3 \frac{\partial t}{\partial \beta} \left(\frac{\partial^2 t}{\partial \alpha \partial \beta} \frac{\partial s}{\partial \beta} - \frac{\partial t}{\partial \beta} \frac{\partial^2 s}{\partial \alpha \partial \beta} \right) \\
&\quad + \frac{2\ \alpha\ (4\alpha^4 + \beta)\ D}{Z\ r_{11}} \frac{\partial t}{\partial \alpha} \frac{\partial r_{11}}{\partial \alpha} - \frac{2\ D}{r_{11}} \frac{\partial t}{\partial \beta} \frac{\partial r_{11}}{\partial \alpha} \\
&\quad - \frac{4\ D}{r_{11}} \frac{\partial t}{\partial \alpha} \frac{\partial r_{11}}{\partial \beta} + 64\alpha^3 \frac{\partial t}{\partial \alpha} \left(\frac{\partial t}{\partial \beta} \frac{\partial^2 s}{\partial \beta^2} - \frac{\partial^2 t}{\partial \beta^2} \frac{\partial s}{\partial \beta} \right) \\
&\quad + 2 \frac{\partial t}{\partial \beta} \left(\frac{\partial t}{\partial \alpha} \frac{\partial^2 s}{\partial \alpha \partial \beta} - \frac{\partial^2 t}{\partial \alpha \partial \beta} \frac{\partial s}{\partial \alpha} \right) + \frac{64\ \alpha^3\ D}{r_{11}} \frac{\partial t}{\partial \beta} \frac{\partial r_{11}}{\partial \beta} \\
&\quad \left. + \frac{\alpha\ (4\alpha^4 + \beta)}{Z} \frac{\partial t}{\partial \alpha} \left(\frac{\partial^2 t}{\partial \alpha^2} \frac{\partial s}{\partial \beta} - \frac{\partial t}{\partial \beta} \frac{\partial^2 s}{\partial \alpha^2} \right) \right) \qquad \text{(A.22)}
\end{aligned}
$$

and

$$
\begin{aligned}
B_{54}(r_{11}, t, s) \ &= \ \frac{C}{D^2\ \Delta\ r_{11}} \frac{\partial t}{\partial \alpha} \left(\frac{\alpha^2\ (16\alpha^4 \beta + 3\beta^2 + 1)}{2\ Z^2} \left(\frac{\partial t}{\partial \alpha} \frac{\partial^2 s}{\partial \alpha \partial \beta} - \frac{\partial^2 t}{\partial \alpha \partial \beta} \frac{\partial s}{\partial \alpha} \right) \right. \\
&\quad + 8 \left(\frac{\partial t}{\partial \beta} \frac{\partial^2 s}{\partial \beta^2} - \frac{\partial^2 t}{\partial \beta^2} \frac{\partial s}{\partial \beta} \right) \\
&\quad + \frac{4\ \alpha\ (4\alpha^4 + \beta)}{Z} \left(\frac{\partial^2 t}{\partial \alpha \partial \beta} \frac{\partial s}{\partial \beta} - \frac{\partial t}{\partial \beta} \frac{\partial^2 s}{\partial \alpha \partial \beta} \right) \\
&\quad + \frac{4\ \alpha\ (4\alpha^4 + \beta)}{Z} \left(\frac{\partial^2 t}{\partial \beta^2} \frac{\partial s}{\partial \alpha} - \frac{\partial t}{\partial \alpha} \frac{\partial^2 s}{\partial \beta^2} \right) \\
&\quad + \frac{3\ \alpha\ (16\alpha^4 \beta + 3\beta^2 + 1)}{2\ Z^2} D \\
&\quad \left. + \frac{\alpha^2\ (16\alpha^4 \beta + 3\beta^2 + 1)\ D}{2\ Z^2\ r_{11}} \frac{\partial r_{11}}{\partial \alpha} \right.
\end{aligned}
$$

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$$-\frac{4\ \alpha\ (4\alpha^4 + \beta)\ D}{Z\ r_{11}}\frac{\partial r_{11}}{\partial \beta}\Bigg).$$

(A.23)

The other elements contain too many terms to be reproduced here explicitly. For example, $A_{64}(r_{11}, t, s)$ and $B_{64}(r_{11}, t, s)$ possess 328 and 532 terms, respectively, and hence we shall refrain from presenting them.

Finally, let us discuss some properties of the solutions of (3.35). If we write

$$A_{44}(r_{11}, t, s) = \frac{C}{D^2\ \Delta\ r_{11}}\frac{\partial t}{\partial \alpha}F(r_{11}, t, s)$$

(A.24)

and

$$B_{54}(r_{11}, t, s) = \frac{C}{D^2\ \Delta\ r_{11}}\frac{\partial t}{\partial \alpha}E(r_{11}, t, s),$$

(A.25)

then $F(r_{11}, t, s)$ and $E(r_{11}, t, s)$ are bilinear and symmetric in t and s

$$F(r_{11}, t, s) = -F(r_{11}, s, t), \quad E(r_{11}, t, s) = -E(r_{11}, s, t).$$

(A.26)

It is thus easy to show that the solutions of $A_{44} = 0$ and $B_{54} = 0$ are allowed to be transformed linearly; $t \to \tilde{t} = at + bs$ and $s \to \tilde{s} = ct + ds$. However, in spite of the non-linear property of B_{44}, A_{64} and B_{64}, we have observed by explicit calculations that the solutions of $B_{44} = 0$, $A_{64} = 0$ and $B_{64} = 0$ are also allowed to be transformed linearly. This implies that these equations may reduce to more fundamental ones.

References

[1] E. Witten, Nucl. Phys. **B268** (1986) 79.

[2] E. Martinec, Phys. Lett. **B171** (1986) 189.

[3] M. Dine and N. Seiberg, Phys. Pev. Lett. **57** (1986) 2625.

[4] P. Candelas, X. de la Ossa, P. Green and L. Parkes, Nucl. Phys. **B359** (1991) 21.

[5] B.R. Greene and M.R. Plesser, Nucl. Phys. **B338** (1990) 15.

[6] D.R. Morrison, Picard-Fuchs equations and mirror maps for hypersurfaces, *in* Essays on Mirror Manifolds, Ed. S.-T. Yau (International Press, Hong Kong, 1992).

[7] A. Font, Nucl. Phys. **B391** (1993) 358.

[8] A. Klemm and S. Theisen, Nucl. Phys. **B389** (1993) 153.

[9] S. Hosono, A. Klemm, S. Theisen and S.-T. Yau, Commun. Math. Phys. **167** (1995) 301; Nucl. Phys. **B433** (1995) 501.

[10] P. Candelas, X. de la Ossa, A. Font S. Katz and D.R. Morrison, Nucl. Phys. **B416** (1994) 481.

[11] P. Candelas, A. Font, S. Katz and D.R. Morrison, Nucl. Phys. **B429** (1994) 626.

[12] S. Hosono and B.H. Lian, GKZ hypergeometric systems and applications to mirror symmetry, preprint hep-th/9602147.

[13] K. Saito, On the periods of Primitive Integrals (Harvard Lecture Notes, 1980); Publ. RIMS, Kyoto University **19**, (1983) 1231; M. Noumi, Tokyo J. Math. **7** (1984) 1.

[14] S. Cecotti and C. Vafa, Nucl. Phys. **B367** (1991) 359.

[15] E. Witten, Commun. Math. Phys. **117** (1988) 353; Commun. Math. Phys. **118** (1988) 411; Nucl. Phys. **B340** (1990) 281.

[16] R. Dijkgraaf, H. Verlinde and E. Verlinde, Nucl. Phys. **B352** (1991) 59.

[17] B. Blok and A. Varchenko, Int. J. Mod. Phys. **A7** (1992) 1467.

[18] E. Verlinde and N.P. Warner, Phys. Lett. **B269** (1991) 96.

[19] W. Lerche, D.-J. Smit and N.P. Warner, Nucl. Phys. **B372** (1992) 87.

[20] A. Klemm, S. Theisen and M.G. Schmidt, Int. J. Mod. Phys. **A7** 25 (1992) 6215.

[21] T. Eguchi, Y. Yamada and S.-K. Yang, Mod. Phys. Lett. **A8** (1993) 1627.

[22] A. Strominger and E. Witten, Commun. Math. Phys. **101** (1985) 341.

[23] P. Berglund, P. Candelas, X. de la Ossa, A. Font, T. Hübsch, D. Jančić and F. Quevedo, Nucl. Phys. **B419** (1994) 352.